

Temperley-Lieb K-matrices

A. Lima-Santos

*Universidade Federal de São Carlos, Departamento de Física
Caixa Postal 676, CEP 13569-905 São Carlos, Brasil*

Abstract

We construct reflection K-matrices of solvable vertex models from representations of the Temperley-Lieb algebras associated with the quantum groups $\mathcal{U}_q(X_n)$ for the affine Lie algebras $X_n = A_1^{(1)}, B_n^{(1)}, C_n^{(1)}$ and $D_n^{(1)}$.

Keywords: Open boundary conditions, Reflection K-matrices.

January 4, 2011

1 Introduction

The Temperley-Lieb (TL) algebra [1] is known to essentially govern the physical properties of a large class of solvable lattice models in two-dimensional statistical mechanics. The association of the TL algebra leads to some equivalence relations among the models as formulated in [2, 3]. From the paper [4] we know the study of the solvable models associated with the TL algebra in the framework of quantum groups.

In [5] we have presented the solutions of the reflection equations for the vertex models from representation of the TL algebra associated with the quantum group $\mathcal{U}_q[sl(2)]$.

Recently, classification of the solutions of the constant reflection equation related to constant TL R -matrices were proposed in [6] and their spectral parameter dependence obtained by Yang-Baxterization.

In this paper we again touch upon the problem of classification of the K -matrix solutions for the TL models, showing that the technical approach used in [5] it is sufficiently general in order to include the solvable vertex models from representations of the TL algebras associated with the quantum groups $\mathcal{U}_q[X_n]$ for $X_n = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}$ and $D_n^{(1)}$.

We have organized this paper as follows. In Section 2 the models are presented, in Section 3 we choose the reflection equations and their solutions. The Section 4 is reserved for the conclusion.

2 The model

From the representation of the TL algebra, one can build solvable vertex models with the R operator defined by

$$R(u) = x_1(u)\mathcal{I} + x_2(u)\mathcal{U}, \quad (2.1)$$

where \mathcal{I} is the identity operator and \mathcal{U} is the TL projector. Here u is the spectral parameter and the anisotropic parameter η is choose so that

$$\begin{aligned} x_1(u) &= \frac{\sinh(\eta - u)}{\sinh \eta}, & x_2(u) &= \frac{\sinh u}{\sinh \eta}, \\ 2 \cosh \eta &= \text{Tr } \mathcal{U}. \end{aligned} \quad (2.2)$$

Setting

$$R_j(u) = 1 \otimes \cdots \otimes 1 \otimes \underbrace{R(u)}_{j, j+1} \otimes 1 \cdots \otimes 1 \quad (2.3)$$

one can show that the Yang-Baxter equation

$$R_{j+1}(u)R_j(u+v)R_{j+1}(v) = R_j(v)R_{j+1}(u+v)R_j(u) \quad (2.4)$$

is valid due to the definition relations of the TL algebra

$$\begin{aligned}
\mathcal{U}_j^2 &= 2 \cosh \eta \mathcal{U}_j \\
\mathcal{U}_j \mathcal{U}_{j \pm 1} &= \mathcal{U}_j \\
\mathcal{U}_i \mathcal{U}_j &= \mathcal{U}_j \mathcal{U}_i \quad |i - j| > 1
\end{aligned} \tag{2.5}$$

For the affine Lie algebras $A_1^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ *i.e.*, the q -deformations of the spin- s representation of $sl(2)$ and the vector representation of $so(2n+1)$, $sp(2n)$ and $so(2n)$, the corresponding Temperley-Lieb projector has the form

$$\mathcal{U} = \sum_{i,j=1}^N \varepsilon(i) \varepsilon(j) q^{-\langle \epsilon_i + \epsilon_j, \bar{\rho} \rangle} e_{i,j} \otimes e_{i',j'} \tag{2.6}$$

where $e_{i,j}$ is the matrix unit ($e_{i,j} v_k = \delta_{j,k} v_i$) and we have used the conjugated index $a' = N + 1 - a$. Here we recall the notation of [4] to introduce a set of orthonormal vectors $\langle \epsilon_i, \epsilon_j \rangle = \delta_{i,j}$, the sign $\varepsilon(i)$ and $\bar{\rho}$, the half-sum of positive roots of the q -deformed affine Lie algebras. In the sequence, let us write explicitly (2.6) for each model

- $A_1^{(1)}$: The $\mathcal{U}_q[sl(2)]$ Temperley-Lieb model

$$N = 2s + 1 \quad (s = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots)$$

$$\varepsilon(i) = (-1)^i$$

$$\bar{\rho} = \frac{1}{2}(\epsilon_1 - \epsilon_2)$$

$$\epsilon_i = (s - i + 1)(\epsilon_1 - \epsilon_2)$$

$$\mathcal{U} = \sum_{i=1}^N \sum_{j=1}^N (-1)^{i+j} q^{i+j-N-1} e_{i,j} \otimes e_{i',j'}$$

$$2 \cosh \eta = [2s + 1] \tag{2.7}$$

- $B_n^{(1)} (n \geq 2)$: The $\mathcal{U}_q[so(2n+1)]$ Temperley-Lieb model

$$N = 2n + 1$$

$$\varepsilon(i \neq n+1) = 1, \quad \varepsilon(n+1) = -1$$

$$\bar{\rho} = \sum_{i=1}^n (n - i + 1/2) \epsilon_i$$

$$\epsilon_{n+1} = 0, \quad \epsilon_k = -\epsilon_{k'} \quad (k = n+2, \dots, N)$$

$$\begin{aligned} \mathcal{U} = & \sum_{i=1}^n \sum_{j=1}^n q^{i+j-N} \mathbf{e}_{i,j} \otimes \mathbf{e}_{i',j'} - \sum_{i=1}^n q^{i-n-1/2} \mathbf{e}_{i,n+1} \otimes \mathbf{e}_{i',n+1} \\ & + \sum_{i=1}^n \sum_{j=n+2}^N q^{i+j-N-1} \mathbf{e}_{i,j} \otimes \mathbf{e}_{i',j'} - \sum_{j=1}^n q^{j-n-1/2} \mathbf{e}_{n+1,j} \otimes \mathbf{e}_{n+1,j'} \\ & + \mathbf{e}_{n+1,n+1} \otimes \mathbf{e}_{n+1,n+1} - \sum_{j=n+2}^N q^{j-n-3/2} \mathbf{e}_{n+1,j} \otimes \mathbf{e}_{n+1,j'} \\ & + \sum_{i=n+2}^N \sum_{j=1}^n q^{i+j-N-1} \mathbf{e}_{i,j} \otimes \mathbf{e}_{i',j'} - \sum_{i=n+2}^N q^{i-n-3/2} \mathbf{e}_{i,n+1} \otimes \mathbf{e}_{i',n+1} \\ & + \sum_{i=n+2}^N \sum_{j=n+2}^N q^{i+j-N-2} \mathbf{e}_{i,j} \otimes \mathbf{e}_{i',j'} \end{aligned}$$

$$2 \cosh \eta = \frac{[2n-1][n+\frac{1}{2}]}{[n-\frac{1}{2}]} \quad (2.8)$$

- $C_n^{(1)}(n \geq 1)$: The $\mathcal{U}_q[sp(2n)]$ Temperley-Lieb model

$$N = 2n$$

$$\varepsilon(i) = 1 \quad (i = 1, \dots, n), \quad \varepsilon(i) = -1 \quad (i = n+1, \dots, N)$$

$$\bar{\rho} = \sum_{i=1}^n (n-i+1) \epsilon_i$$

$$\epsilon_k = -\epsilon_{k'} \quad (k = n+1, \dots, N)$$

$$\begin{aligned} \mathcal{U} = & \sum_{i=1}^n \sum_{j=1}^n q^{i+j-N-2} \mathbf{e}_{i,j} \otimes \mathbf{e}_{i',j'} - \sum_{i=1}^n \sum_{j=n+1}^N q^{i+j-N-1} \mathbf{e}_{i,j} \otimes \mathbf{e}_{i',j'} \\ & - \sum_{i=n+1}^N \sum_{j=1}^n q^{i+j-N-1} \mathbf{e}_{i,j} \otimes \mathbf{e}_{i',j'} + \sum_{i=n+1}^N \sum_{j=n+1}^N q^{i+j-N} \mathbf{e}_{i,j} \otimes \mathbf{e}_{i',j'} \end{aligned}$$

$$2 \cosh \eta = \frac{[n][2n+2]}{[n+1]} \quad (2.9)$$

- $D_n^{(1)}(n \geq 3)$: The $\mathcal{U}_q[so(2n)]$ Temperley-Lieb model

$$N = 2n$$

$$\varepsilon(i) = 1$$

$$\bar{\rho} = \sum_{i=1}^n (n-i) \epsilon_i$$

$$\epsilon_k = -\epsilon_{k'} \quad (k = n+1, \dots, N)$$

$$\begin{aligned} \mathcal{U} = & \sum_{i=1}^n \sum_{j=1}^n q^{i+j-N} \mathbf{e}_{i,j} \otimes \mathbf{e}_{i',j'} + \sum_{i=1}^n \sum_{j=n+1}^N q^{i+j-N-1} \mathbf{e}_{i,j} \otimes \mathbf{e}_{i',j'} \\ & + \sum_{i=n+1}^N \sum_{j=1}^n q^{i+j-N-1} \mathbf{e}_{i,j} \otimes \mathbf{e}_{i',j'} + \sum_{i=n+1}^N \sum_{j=n+1}^N q^{i+j-N-2} \mathbf{e}_{i,j} \otimes \mathbf{e}_{i',j'} \end{aligned}$$

$$2 \cosh \eta = \frac{[n][2n-2]}{[n-1]} \quad (2.10)$$

Here we remark the quantum number notation

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \quad (2.11)$$

We also have to consider the permuted operator $\mathcal{R} = PR$ which is regular satisfying PT-symmetry, unitarity and crossing symmetry

$$\begin{aligned} \mathcal{R}_{12}(0) &= P, \\ \mathcal{R}_{12}^{t_1 t_2}(u) &= P \mathcal{R}_{12}(u) P = \mathcal{R}_{21}(u), \\ \mathcal{R}_{12}(u) \mathcal{R}_{21}(-u) &= x_1(u) x_1(-u) I, \\ \mathcal{R}_{21}(u) &= \kappa(V \otimes 1) \mathcal{R}_{12}^{t_2}(-u - \rho)(V \otimes 1)^{-1} \end{aligned} \quad (2.12)$$

where $\rho = -\eta$ is the crossing parameter, $\kappa = (-1)^{2s}$ for $A_1^{(1)}$, $\kappa = -1$ for $C_n^{(1)}$ and $\kappa = 1$ for $B_n^{(1)}$ and $D_n^{(1)}$. The crossing matrices V for the TL models are specified by

$$V_{i,j} = \varepsilon(i) q^{-\langle \epsilon_i, \rho \rangle} \delta_{i',j} \quad (2.13)$$

and P is the permutation matrix

$$P = \sum_{i,j=1}^N \mathbf{e}_{i,j} \otimes \mathbf{e}_{j,i} \quad (2.14)$$

3 The reflection matrices

Let us to start the search for boundary integrable models through solutions of the boundary Yang–Baxter equation [7, 8] where the boundary weights follow from K matrices which satisfy a pair of equations, namely the reflection equation

$$\mathcal{R}_{12}(u-v) K_1^-(u) \mathcal{R}_{12}^{t_1 t_2}(u+v) K_2^-(v) = K_2^-(v) \mathcal{R}_{12}(u+v) K_1^-(u) \mathcal{R}_{12}^{t_1 t_2}(u-v) \quad (3.1)$$

and the dual reflection equation

$$\begin{aligned} \mathcal{R}_{12}(-u+v) (K_1^+)^{t_1}(u) M_1^{-1} \mathcal{R}_{12}^{t_1 t_2}(-u-v-2\rho) M_1 (K_2^+)^{t_2}(v) = \\ (K_2^+)^{t_2}(v) M_1 \mathcal{R}_{12}(-u-v-2\rho) M_1^{-1} (K_1^+)^{t_1}(u) \mathcal{R}_{12}^{t_1 t_2}(-u+v). \end{aligned} \quad (3.2)$$

In this case duality supplies a relation between K^- and K^+ [9]

$$K^+(u) = K^-(-u - \rho)^t M, \quad M = V^t V \quad (3.3)$$

Here t denotes transposition and t_i denotes transposition in the i -th space. V is the crossing matrix and ρ the crossing parameter, both being specific to each model [10].

3.1 Diagonal reflection matrices

Taking into account only the diagonal form for the K^- matrices, one can substitute

$$K^-(u) = \sum_{i=1}^N k_{i,i}(u) e_{i,i} \doteq \text{diag}(k_{1,1}(u), \dots, k_{N,N}(u)) \quad (3.4)$$

and $\mathcal{R}(u) = P[x_1(u)\mathcal{I} + x_2(u)\mathcal{U}]$ into (3.1), in order to get N^2 functional equations for the $k_{i,i}$ elements, many of them not independent equations. In order to solve these functional equations, we shall proceed as follows. First we consider the (i, j) component of the matrix equation (3.1). By differentiating it with respect to v and taking $v = 0$, we get algebraic equations involving the single variable u and N parameters

$$\beta_{i,i} = \left. \frac{dk_{i,i}(v)}{dv} \right|_{v=0}, \quad i, = 1, 2, \dots, N \quad (3.5)$$

The reflection equations are solved when we find all matrix elements $k_{j,j}(u)$, $j = 2, \dots, N$ as function of $k_{1,1}(u)$, provided that the parameters $\beta_{j,j}$ satisfy $\frac{1}{2}(N-1)(N-2)$ constraint equations of the type

$$(\beta_{N,N} - \beta_{i,i})(\beta_{N,N} - \beta_{j,j})(\beta_{j,j} - \beta_{i,i}) = 0 \quad (i \neq j \neq N) \quad (3.6)$$

From (3.6) we can find diagonal K^- matrix solutions with only two type of entries. Let us normalize one of them to be equal to 1 such that the other entry is given by

$$k_{p,p}(u) = -\frac{\beta_{p,p}x_2(u)[\Delta_1x_2(u) + x_1(u)] + 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]}{\beta_{p,p}x_2(u)[\Delta_kx_2(u) + x_1(u)] - 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]} \quad (3.7)$$

where $\Delta_1 + \Delta_k = 2 \cosh \eta$.

Here we notice that the constraint equations (3.6) are the same for all TL models and the model differences come from (3.7) by the quantum group dependence of the terms Δ_1 and Δ_k .

Identifying the diagonal positions with the powers of q , *i.e.* $(q^{-2\langle \epsilon_1, \vec{\rho} \rangle}, q^{-2\langle \epsilon_2, \vec{\rho} \rangle}, \dots, q^{-2\langle \epsilon_N, \vec{\rho} \rangle})$, one can see that Δ_1 is the sum of the powers of q corresponding to the positions of the entries 1 and Δ_k is the sum of the power of q corresponding to the positions of the entries $k_{p,p}(u)$.

Denoting the diagonal solutions by $\mathbb{K}_{\mathbf{a}}^{[r]}$ where $\mathbf{a} = (a_1, a_2, \dots, a_N)$ with $a_i = 0$ if $k_{i,i}(u) = 1$ or $a_i = 1$ if $k_{i,i}(u) = k_{p,p}(u)$ and r is the number of the entries $k_{p,p}(u)$ distributed on diagonal positions and p being the first position with the entry different from 1. Thus, we have counted

$$N = \sum_{r=1}^{N-1} \frac{N!}{r!(N-r)!} \quad (3.8)$$

for the number of diagonal K^- matrix solutions with one free parameter.

The solutions of the dual reflection equation (3.2) are obtained by the isomorphism (3.3) where the M matrices for the TL models are specified by

$$M_{i,j} = q^{-2\langle \epsilon_i, \vec{\rho} \rangle} \delta_{i,j} \quad (3.9)$$

Let us work explicitly with this description for $N = 4$. There are four matrices of the type $\mathbb{K}_{\mathbf{a}}^{[1]}$

$$\begin{aligned} \mathbb{K}_{(1,0,0,0)}^{[1]} &= \text{diag}(k_{11}(u), 1, 1, 1), & \mathbb{K}_{(0,1,0,0)}^{[1]} &= \text{diag}(1, k_{22}(u), 1, 1), \\ \mathbb{K}_{(0,0,1,0)}^{[1]} &= \text{diag}(1, 1, k_{33}(u), 1), & \mathbb{K}_{(0,0,0,1)}^{[1]} &= \text{diag}(1, 1, 1, k_{44}(u)). \end{aligned} \quad (3.10)$$

In particular, for $\mathbb{K}_{(1,0,0,0)}^{[1]}$, the entrie $k_{11}(u)$ is given by (3.7) with

$$\begin{aligned} \Delta_1 &= q^{-2\langle \epsilon_2, \vec{\rho} \rangle} + q^{-2\langle \epsilon_3, \vec{\rho} \rangle} + q^{-2\langle \epsilon_4, \vec{\rho} \rangle}, \\ \Delta_k &= q^{-2\langle \epsilon_1, \vec{\rho} \rangle}. \end{aligned} \quad (3.11)$$

There are six matrices of the type $\mathbb{K}_{\mathbf{a}}^{[2]}$

$$\begin{aligned} \mathbb{K}_{(1,1,0,0)}^{[2]} &= \text{diag}(k_{11}(u), k_{11}(u), 1, 1), & \mathbb{K}_{(1,0,1,0)}^{[2]} &= \text{diag}(k_{11}(u), 1, k_{11}(u), 1), \\ \mathbb{K}_{(1,0,0,1)}^{[2]} &= \text{diag}(k_{11}(u), 1, 1, k_{11}(u)), & \mathbb{K}_{(0,1,1,0)}^{[2]} &= \text{diag}(1, k_{22}(u), k_{22}(u), 1), \\ \mathbb{K}_{(0,1,0,1)}^{[2]} &= \text{diag}(1, k_{22}(u), 1, k_{22}(u)), & \mathbb{K}_{(0,0,1,1)}^{[2]} &= \text{diag}(1, 1, k_{33}(u), k_{33}(u)). \end{aligned} \quad (3.12)$$

In particular, for $\mathbb{K}_{(0,0,1,1)}^{[2]}$, the entrie $k_{33}(u)$ is given by (3.7) with

$$\begin{aligned} \Delta_1 &= q^{-2\langle \epsilon_1, \vec{\rho} \rangle} + q^{-2\langle \epsilon_2, \vec{\rho} \rangle}, \\ \Delta_k &= q^{-2\langle \epsilon_3, \vec{\rho} \rangle} + q^{-2\langle \epsilon_4, \vec{\rho} \rangle} \end{aligned} \quad (3.13)$$

There are more four matrices of the type $\mathbb{K}_{\mathbf{a}}^{[3]}$

$$\begin{aligned} \mathbb{K}_{(0,1,1,1)}^{[3]} &= \text{diag}(1, k_{22}(u), k_{22}(u), k_{22}(u)), & \mathbb{K}_{(1,0,1,1)}^{[3]} &= \text{diag}(k_{11}(u), 1, k_{11}(u), k_{11}(u)), \\ \mathbb{K}_{(1,1,0,1)}^{[3]} &= \text{diag}(k_{11}(u), k_{11}(u), 1, k_{11}(u)), & \mathbb{K}_{(1,1,1,0)}^{[3]} &= \text{diag}(k_{11}(u), k_{11}(u), k_{11}(u), 1). \end{aligned} \quad (3.14)$$

For $\mathbb{K}_{(1,1,1,0)}^{[3]}$ we have

$$\begin{aligned} \Delta_1 &= q^{-2\langle \epsilon_4, \vec{\rho} \rangle} \\ \Delta_k &= q^{-2\langle \epsilon_1, \vec{\rho} \rangle} + q^{-2\langle \epsilon_2, \vec{\rho} \rangle} + q^{-2\langle \epsilon_3, \vec{\rho} \rangle} \end{aligned} \quad (3.15)$$

The TL models for $N = 4$ are $A_1^{(1)}(s = \frac{3}{2})$ and $C_2^{(1)}$ where $\Delta_1 + \Delta_k$ is $q^{-3} + q^{-1} + q + q^3$ and $q^{-4} + q^{-2} + q^2 + q^4$, respectively.

Finally, we notice that the difference between the entries of the several $\mathbb{K}_{\mathbf{a}}^{[r]}$ matrices come from the partitions of $2 \cosh \eta$ such that

$$\Delta_1 + \Delta_k = \sum_{k=1}^N q^{-2 \langle \epsilon_k, \tilde{\rho} \rangle}. \quad (3.16)$$

Moreover, the equivalence between the $\mathbb{K}_{\mathbf{a}}^{[r_1]}$ and $\mathbb{K}_{\mathbf{a}}^{[r_2]}$ matrices with $r_1 + r_2 = N$ are obtained by the transformation $q \rightarrow q^{-1}$.

3.2 Non-diagonal reflection matrices

Now the normal K^- -matrix has the form

$$K^-(u) = \sum_{i,j=1}^N k_{i,j}(u) e_{i,j} \quad (3.17)$$

and from (3.1) we will have N^4 functional equations for the $k_{i,j}$ elements and N^2 parameters

$$\beta_{i,j} = \left. \frac{dk_{i,j}(v)}{dv} \right|_{v=0}, \quad (i, j = 1, \dots, N) \quad (3.18)$$

The normal condition means $k_{i,j}(0) = \delta_{i,j}$.

Analyzing the reflection equations one can see that several exist involving only two non-diagonal elements. They can be solved by the relations

$$k_{i,j}(u) = \frac{\beta_{i,j}}{\beta_{1,N}} k_{1,N}(u) \quad (i \neq j = \{1, 2, \dots, N\}) \quad (3.19)$$

We thus left with several equations involving two diagonal elements and $k_{1,N}(u)$. Such equations are solved by the relations

$$k_{i,i} = k_{1,1}(u) + (\beta_{i,i} - \beta_{1,1}) \frac{k_{1,N}(u)}{\beta_{1,N}} \quad (i = 2, \dots, N). \quad (3.20)$$

Finally, we can use the equation $(1, N)$ in order to find the element $k_{1,1}(u)$:

$$\begin{aligned} k_{1,1}(u) = & \frac{k_{1,N}(u)}{\beta_{1,N}[x_2(u) \cosh \eta + x_1(u)]} \left\{ \frac{x_1(u)x_2'(u) - x_1'(u)x_2(u)}{x_2(u)} \right. \\ & \left. - \frac{1}{2}x_1(u)(\beta_{N,N} - \beta_{1,1} + \Psi_{1,N}) - \frac{1}{2}x_2(u) \sum_{j=2}^N (\beta_{j,j} - \beta_{1,1}) q^{-\langle 2\epsilon_j, \tilde{\rho} \rangle} \right\} \end{aligned} \quad (3.21)$$

where $x_i'(u) = dx_i(u)/du$, $i = 1, 2$ and

$$\Psi_{1,N} = \frac{1}{\beta_{1,N}} \sum_{k=2}^{N-1} \beta_{1,k} \beta_{k,N} \quad (3.22)$$

At this point, we have written all matrix elements of $K^-(u)$ in terms of $k_{1,N}(u)$ which is an arbitrary function satisfying the normal condition.

Again, we notice that these results are valid for all TL models, here differentiated by the factor $q^{-\langle 2\epsilon_j, \tilde{\rho} \rangle}$ in the diagonal elements of the K^- -matrices.

Now, substituting these expressions into the remained equations (i, j) of (3.1), we are left with several constraint equations involving the $\beta_{i,j}$ parameters. The existence of many free parameters turn out that it is a very formidable task.

In order to proceed with this task we follow the steps of [5] where we have introduced two new objects instead of working directly with the $\beta_{i,j}$ parameters

$$\Psi_{i,j} = \frac{1}{\beta_{i,j}} \sum_{k \neq i,j} \beta_{i,k} \beta_{k,j} \quad \text{and} \quad \Theta_{i,i} = \sum_{k \neq i} \beta_{i,k} \beta_{k,i}. \quad (3.23)$$

After we rewrite the all constraint equations in terms of $\Psi_{i,j}$ and $\Theta_{i,i}$, we can easily find the diagonal parameters,

$$\begin{aligned} \beta_{i,i} &= \beta_{1,1} + \Psi_{1,N} - \Psi_{i,N}, & (i = 2, 3, \dots, N-1) \\ \beta_{N,N} &= \beta_{1,1} + \Psi_{1,N-1} - \Psi_{N-1,N}. \end{aligned} \quad (3.24)$$

The parameter $\beta_{1,1}$ is fixed by the normal condition.

All constraint equations are now substituted by $\frac{1}{2}(N-1)N$ symmetric relations

$$\Psi_{j,i} = \Psi_{i,j}, \quad (j > i) \quad (3.25)$$

$2(N-3)$ relations of the type

$$\begin{aligned} \Psi_{2,j} &= \Psi_{2,3} + \Psi_{1,j} - \Psi_{1,3}, & (j = 4, \dots, N), \\ \Psi_{3,j} &= \Psi_{2,3} + \Psi_{1,j} - \Psi_{1,2}, & (j = 4, \dots, N). \end{aligned} \quad (3.26)$$

and $\frac{1}{2}(N-4)(N-3)$ relations involving the $\Psi_{i,j}$

$$\Psi_{i,j} = \Psi_{1,i} + \Psi_{1,j} - \Psi_{1,2} - \Psi_{1,3}, \quad (i = 4, \dots, N-1, \quad j = i+1, \dots, N) \quad (3.27)$$

Moreover, there are $2(N-3)$ relations involving the diagonal $\beta_{k,k}$ parameters, $\Psi_{1,N}$ and the $\Theta_{j,j}$,

$$\begin{aligned} \Theta_{j,j} &= \Theta_{N,N} + (\beta_{N,N} - \beta_{j,j})(\beta_{j,j} - \beta_{1,1} - \Psi_{1,N}), & (j = 2, 3, \dots, N-1), \\ \Theta_{j',j'} &= \Theta_{1,1} + (\beta_{1,1} - \beta_{j',j'})(\beta_{j',j'} - \beta_{N,N} - \Psi_{1,N}), & (j = 2, 3, \dots, N-1), \\ \Theta_{N,N} &= \Theta_{1,1} - (\beta_{1,1} - \beta_{N,N})\Psi_{1,N}, \end{aligned} \quad (3.28)$$

where $j' = N+1-j$.

As one can see from these expressions, they are independent of the quantum group parameter q . Consequently, the same for all TL models.

From these relations one can account $N^2 - 3$ constraint equations but, after the all substitutions, we only need to look at the symmetric relations (3.25). We can solve simultaneously these $\frac{1}{2}(N-1)N$ symmetric relations in terms of the $\beta_{i,j}$, taking into account the solution with smallest number of fixed parameters $\beta_{i,j}$ which also satisfy the constraint equations (3.28). Following this procedure we have fixed $\frac{1}{2}[(N-1)^2 - 2]$ parameters $\beta_{i,j}$ for the models with N odd and $\frac{1}{2}[(N-1)^2 - 3]$ parameters $\beta_{i,j}$ for the models with N even in addition to the $N-1$ diagonal parameters $\beta_{i,i}$ already fixed by (3.24). It means that we have found the normal reflection K -matrices for the TL models. These are solution of (3.1) with $\frac{1}{2}(N^2 + 1)$ free parameters if N odd and with $\frac{1}{2}(N^2 + 2)$ free parameters if N even.

The solution of the dual equation (3.2) is obtained by the isomorphism (3.3) with $\rho = -\eta$ and the matrix M given by (3.9).

In [5] these relations are solved explicitly for the first values of N (spin $s = 1/2, 1, 3/2, 2$). Moreover, based on the general solutions several specific cases of particular interest are presented.

4 Conclusion

In this work we have presented the normal N by N reflection K -matrices for the vertex models built from representations of the TL algebras associated with quantum group $\mathcal{U}_q[X_n]$ for the affine Lie algebras $X_n = sl(2), so(2n+1), sp(2n)$ and $so(2n)$. Our findings can be summarized into two classes of general solutions depending of the parity of N . An important property of these solutions it is their very large number of free parameters [5, 6].

These results pave the way to construct, solve and study physical properties of the underlying quantum spin chains with open boundaries, generalizing the previous efforts made for the case of periodic boundary conditions [11, 12]. Although we do not know the Algebraic Bethe Ansatz for $N \neq 2$, even for the periodic cases, we expect that the coordinate Bethe ansatz solution of the Temperley-Lieb models constructed from diagonal solutions presented here can be obtained by adapting the results of [13] and the algebraic-functional method presented in [14] may be a possibility to treat the non-diagonal cases. We expect the results presented here to motivate further developments on the subject of integrable open boundaries for the Temperley-Lieb vertex models based on q -deformed superalgebras [15].

Acknowledgment: This work was supported in part by Fundação de Amparo à Pesquisa do Estado de São Paulo-FAPESP-Brasil and by Conselho Nacional de Desenvolvimento-CNPq-Brasil.

References

- [1] Temperley H N V and Lieb E H, 1971 *Proc. R. Soc.* **A 322** 251.
- [2] Baxter R J, 1982 *Exactly solved models in statistical mechanics* (London: Academic Press)
- [3] Martin P, 1991 *Potts models and related problems in statistical mechanics* (Singapore: World Scientific)
- [4] Batchelor M T and Kuniba A, 1991 *J.Phys. A: Math. Gen.* **24** 2599.
- [5] Lima-Santos A, 2010 *On the $U_q[sl(2)]$ Temper-Lieb reflection matrices*, ArXiv:1011.2891v1 [nlin.SI]
- [6] Avan J, Kulish P and Rollet G, 2010 *Reflection K-matrices related to Temperley-Lieb R-matrices*, ArXiv:1012.3012v1 [nlin.SI]
- [7] Cherednik I V, 1984 *Theor. Math. Phys.* **61** 977
- [8] Sklyanin E K, 1988 *J. Phys A: Math. Gen.* **21** 2375
- [9] L. Mezincescu L and Nepomechie R I, 1991 *Int. J. Mod. Phys.* **A 6** 5231; Mezincescu L, Nepomechie R I, 1992 *Int. J. Mod. Phys.* **A 7** 5657 (Addendum)
- [10] Bazhanov V V, 1987 *Commun. Math. Phys.* **113** 47
- [11] Köberle K and Lima-Santos A, *J. Phys. A: Math. Gen.* **29** (1996) 519
- [12] Lima-Santos A, 1988 *Nucl. Phys.* **B 522** [FS] 503
- [13] Lima-Santos A and Ghiotto R C T, 1998 *J. Phys. A: Math. Gen.* **31** 505
- [14] Galleas W, 2008 *Nucl. Phys.* **B 790** 524
- [15] Zhang R B, 1991 *J. Math. Phys.* **32** 2605